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# TWO PION BOUND STATES IN A HOT PION GAS

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## Abstract

The occurrence of poles in the $2\pi$ propagator in a hot pion medium is studied. The domain of non–trivial solutions to the corresponding bosonic gap equation is investigated as a function of the chemical potential and temperature of the gas.

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When two heavy nuclei collide at very high energy, the matter that is created differs qualitatively from that traditionally studied in nuclear and elementary particle physics. In the initial stages of the collision, copious production of gluons and quarks in a large volume leads to a rapid increase in the entropy, and the distinct possibility of a new phase of matter characterized by deconfined partonic degrees of freedom. One therefore hopes that relativistic heavy ion experiments at Brookhaven and CERN will provide some insight into the structure of the QCD vacuum, deconfinement, and chiral symmetry restoration.

The hot transient system eventually evolves into a gas of hadrons at high energy densities, whose interactions may well obscure many potential signals of the deconfined phase. To study this problem, extensive use has been made of sequential scattering models, such as hadronic cascades [1, 2, 3]. In principle, these trace the evolution of the system from hadronization to freeze–out, and can assign features in the data to the very early dense phase. For example, calculations using the Boltzmann equation with bose statistics to simulate heavy ion collisions at beam energies of 200 $A \cdot \mathrm{GeV}$ [3, 4] show that if the initial pion density is large enough, the low transverse momentum component of their spectra will be enhanced during the evolution of the pion gas [5]. Implicit in these studies is a thermalization rate governed by free space $\pi$–$\pi$ interactions. However, recent investigations [6, 7] of the effect of finite phase space occupation on intermediate states suggest that the t–matrix may well be suppressed within the environment of a relativistic heavy ion collision (RHIC) [8]. This quantitatively affects the conclusions one may draw from such simulations, and points directly to the need to better understand hadronic properties and interactions in RHICs.

Here we wish to reconsider $\pi$–$\pi$ interactions in the presence of a dense and hot pion gas along the lines of our previous approach [6]. We shall address in some detail the question of pion pair formation in the medium. As we shall see, the in–medium $2\pi$ propagator exhibits a pole between a lower and an upper critical temperature, signaling a possible instability with respect to pion pair formation.

To simplify the solution of the $2\pi$ propagator, we consider a separable model for the free space $\pi$–$\pi$ interaction [9], in which the $2\pi$ state couples to an intermediate $\sigma$ $(\ell=0, I=0)$ or $\rho$ $(\ell=1, I=1)$ meson. The corresponding scattering matrix may be derived from the following effective Hamiltonian

$$
\begin{aligned}
K &= H - \mu\,(N_\pi - 2\,N_\alpha) \\
&= \sum_k (\omega_k - \mu)\, b_k^\dagger b_k + \sum_a (\Omega_a - 2\mu)\, \sigma_a^\dagger \sigma_a \\
&\qquad + \frac{1}{2} \sum_{k_1 k_2 a} \left\{ b_{k_1} b_{k_2}\, (\sigma_a^\dagger + \sigma_{-a})\, \langle a |\, W\, | k_1 k_2 \rangle + \text{h.c.} \right\} \;, \qquad (1)
\end{aligned}
$$

where the $b_k$ $(b_k^\dagger)$ are the pion annihilation (creation) operators labeled by momentum $\vec{k}$ with energy $\omega_k = \sqrt{\vec{k}^{\,2}+m_\pi^2}$. The $\sigma_a$ $(\sigma_{-a}^\dagger)$ are the corresponding operators for the intermediate meson labeled by $a=(\alpha,\vec{P})$ $(-a=(\alpha,-\vec{P}))$, where $\alpha=(\ell,I)$ are the spin–isospin quantum numbers. The state's energy is $\Omega_a = \sqrt{\vec{P}^{\,2}+M_\alpha^2}$, where $\vec{P}=\vec{k}_1+\vec{k}_2$ is the total momentum, and $M_\alpha$ the mass of the exchanged meson. We chose a Yukawa coupling

$$
\begin{aligned}
\langle a |\, W\, | k_1 k_2 \rangle &= \sqrt{2}\, (2\Omega_a 2\omega_{k_1} 2\omega_{k_2})^{-1/2}\, V_\alpha(k) \qquad (2) \\
V_\alpha(k) &= 4\pi\, g_\alpha\, \omega_k \sqrt{M_\alpha}\, v_\alpha(k) \qquad (3) \\
v_\alpha(k) &= \frac{(k/k_\alpha)^\ell}{[1+(k/k_\alpha)^2]^{\ell+1}} \quad , \qquad (4)
\end{aligned}
$$

where $k$ is the relative momentum of a pion in the c.m. frame of the pion pair with total momentum $\vec{P}$. The constants $g_\alpha$, $M_\alpha$ and $k_\alpha$ are chosen as in Ref. [6], and thus our Hamiltonian reproduces the $\pi$–$\pi$ phase shifts in free space.

The introduction of a chemical potential in the effective Hamiltonian may be justified as follows: There is considerable evidence that the pion distribution at freeze-out in RHICs is near local thermal equilibrium, but not in chemical equilibrium. Since the $\pi$–$\pi$ interaction is essentially elastic up to 1 GeV, in the later stages of the evolution the inelastic collision rate is insufficient to achieve full equilibrium. As the system continues to expand toward freeze–out, the dominant number conserving processes still thermalize the system, but a non–zero chemical potential is built up [3, 4].

At a formal level in a field theoretical scheme it is not possible to define a total pion number operator from the canonical pion field. In the context of our semi–relativistic effective Hamiltonian the total pion number $\sum_i b^\dagger_{i\vec{k}} b_{i\vec{k}}$ ($i$ is the isospin index) does not commute with $H$. It is, however, conserved on time scales larger than the life–time of the intermediate meson: the basic process is two pion annihilation to an intermediate $\sigma$ or $\rho$, which subsequently decays back into two pions. In this spirit, we introduce a chemical potential $\mu$ for the pion. The uncorrelated (causal) $2\pi$ propagator is then given by (at $\vec{P}=0$)

$$G^T_{\pi\pi}(E,k) \;=\; \left\{ \frac{1}{E-2(\omega_k-\mu)+i\eta} - \frac{1}{E+2(\omega_k-\mu)+i\eta} \right\} \coth\frac{\omega_k-\mu}{2T} \quad . \tag{5}$$

Using the $K$–Hamiltonian Eq. (1), we may rederive the in–medium $\sigma$–meson propagator, which, after Fourier transformation, reads

$$\begin{aligned} D^{-1}_\alpha(E) \;=\; E^2 \;&-\; (M_\alpha - 2\mu)^2 \\ &-\; 2(M_\alpha - 2\mu)\,\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\; |\langle 0|\,W\,|k,-k\rangle|^2\; G^T_{\pi\pi}(E,k)\ . \end{aligned} \tag{6}$$

The pole condition $D^{-1}_\sigma(E)=0$ for the existence of bound $2\pi$ states (with $\vec{P}=0$) in the sigma channel ($\ell=0, I=0$) follows from Eq. (6) with (2) and (5):

$$\begin{aligned} \frac{1}{g^2_\sigma} \;&=\; 2(M_\sigma-2\mu)\int_0^\infty dk\, k^2\, \frac{v^2_\sigma(k)}{E^2-(M_\sigma-2\mu)^2}\, \frac{4(\omega_k-\mu)}{E^2-4(\omega_k-\mu)^2}\, \coth\frac{\omega_k-\mu}{2T} \\ &\equiv\; F_{\mu,T}(E) \quad . \end{aligned} \tag{7}$$

In Fig. 1 we show the function $F_{\mu,T}(E)$ for a fixed pion chemical potential of 100 MeV, at five different temperatures (solid lines). These curves intersect with $1/g^2_\sigma \approx 486$ MeV (horizontal dashes) below energies of $2(m_\pi-\mu)$ (the bound state domain) if the temperature lies between a lower and an upper critical value of $T_0 \approx 80$ MeV ($E_{\rm pole}=2(m_\pi-\mu)$) and $T_c \approx 137$ MeV ($E_{\rm pole}=0$), respectively.

Several observations can be made here: (i) One always obtains a pole if the temperature lies between $T_0(\mu)$ (pole at $E=2(m_\pi-\mu)$) and $T_c(\mu)$ (pole at $E=0$). Thus, at fixed $\mu$, no matter how weak the potential $g_\sigma$ is (provided it is attractive in the neighborhood of the $2\pi$ threshold), one always obtains a pole in a range of

sufficiently high temperatures (for fermions at a sufficiently low temperature). In practice, $T_0$ and $T_c$ will exceed sensible values for pions as soon as $\mu$ drops below $\sim$ 100 MeV, since they are increasing functions of $\mu$; (ii) for fixed $g_\sigma$, the pole position shifts downward with increasing temperature (for fermions: pole position moves up with increasing temperature); and (iii) the pole position also shifts down with increasing $g_\sigma$ (as for fermions). As a function of temperature, we therefore see a behavior for bosons opposite to that for fermions.

The fact that increasing temperature reinforces the binding is somewhat counterintuitive, but it is an immediate consequence of the coth factor associated with bose statistics in Eq. (7) (fermions: tanh), which enhances the medium density, effectively reinforcing the 2–body matrix element for increasing temperature. However, the bound state energy drops to zero at a certain critical temperature $T_c(\mu)$. For $T > T_c$, a real solution ceases to exist. As usual, such a zero eigenvalue signals a phase transition point. We shall see shortly that for $T \gtrsim T_c$ the system has become so dense that pairs (corresponding to the pole) begin to obstruct each other and dissolve in the medium.

In analogy with the Cooper pole in fermion scattering, a pole in the two boson propagator also implies a non–trivial solution to the corresponding "gap equation." It is straightforward to derive such an equation from the Hamiltonian Eq. (1)

$$\Delta_k \;=\; 8\pi^2 g_\sigma^2 \int \frac{d^3q}{(2\pi)^3}\, \frac{v_\sigma(k)\, v_\sigma(q)}{M_\sigma - 2\mu}\, \frac{\Delta_q}{2E_q}\, \coth\frac{E_q}{2T} \quad . \tag{8}$$

where $v_\sigma$ is given by Eq. (4), and the boson quasiparticle energy is $E_k^2 = (\omega_k - \mu)^2 - \Delta_k^2$. The ansatz $\Delta_k = \delta\, v_\sigma(k)$ reduces Eq. (8) to a non–linear equation for the gap strength $\delta$.

In spite of the formal similarities of Eq. (8) with the corresponding fermionic gap equation, there are important differences: For bosons, the $\Delta_k^2$ is subtracted in $E_k$ (fermions: added), and the temperature factor is a hyperbolic cotangent (fermions: tanh). We have seen previously that at sufficiently high temperature ($> T_0(\mu)$) there exists a pole in the $2\pi$ propagator, *i.e.*, finite $T$ favors binding, contrary to the fermion case. As we shall see, a similar feature occurs for the gap. At fixed $\mu$, above

a temperature $T_0'(\mu)$ a finite gap appears in the spectrum, which disappears again beyond a critical temperature which will turn out to be $T_c(\mu)$. At temperatures larger than $T_c$, the medium density increases to the point where the pairs dissolve (*i.e.* $\delta = 0$).

Let us now study the domain of existence of a real, finite value of $\Delta$ in the $\mu$–$T$ plane. Rewriting the gap equation (8) as

$$\begin{aligned} \frac{1}{g_\sigma^2} &= 2\int_0^\infty dk\, k^2\, \frac{v_\sigma^2(k)}{M_\sigma - 2\mu}\, \frac{1}{\sqrt{(\omega_k - \mu)^2 - \delta^2 v_\sigma^2(k)}}\, \coth \frac{\sqrt{(\omega_k - \mu)^2 - \delta^2 v_\sigma^2(k)}}{2T} \\ &\equiv G_{\mu,T}(\delta) \quad , \end{aligned} \tag{9}$$

it is clear that for fixed $\mu$ and $T$, $G_{\mu,T}(\delta)$ increases monotonically from $\delta = 0$ to $\delta_{\max} = m_\pi - \mu$. Further, for fixed $\mu$, $G_{\mu,T}$ also increases with temperature. This behavior is shown in Fig. 2 for $\mu = 100$ MeV and various values of $T$ (solid lines). The pair potential $\delta$ is given by the intersection of the solid lines with the horizontal dashed line. With decreasing temperature, it varies from zero at $T = T_c$ to $\delta_{\max} \equiv m_\pi - \mu$ at $T = T_0'$. In Fig. 3 we show the gap strength $\delta = \Delta_k / v_\sigma(k)$ as a function of $m_\pi - \mu$. For a given temperature, $\delta$ vanishes at some upper critical chemical potential, and increases to $\delta_{\max}$ at a lower critical chemical potential.

The critical temperature $T_c$ for which the gap vanishes is determined by linearizing Eq. (8), *i.e.*, setting $\delta = 0$ in Eq. (9). The resulting equation is identical to the pole equation (7) at $E = 0$, and thus the gap and the bound state both disappear at the same critical temperature $T_c$. Eq. (7) is equivalent to what is known in the fermion case as the particle–particle RPA. Thus we see that in the boson case, too, one has the well known property that a phase transition to a pair condensate is signaled by the appearance of a zero eigenvalue in the RPA eigenfrequencies (Thouless criterion) [11].

For fixed $\mu$, as $T$ decreases, $\delta$ increases (see Fig. 2), until it reaches a maximum value of $m_\pi - \mu$ at $T = T_0'$. At this point $E_{k=0} = 0$, and the gap in the excitation spectrum has ceased to exist. Any further lowering of the temperature would yield a purely imaginary quasiparticle energy. In order to understand what happens, we

investigate the pion momentum distribution

$$n_k \;=\; \langle b_k^\dagger b_k \rangle \;=\; \frac{1}{\exp(E_k/T)\;-\;1} \;+\; v_k^2 \,\coth\frac{E_k}{2T} \tag{10}$$

where $v_k^2 \;=\; [\,(\omega_k - \mu)/(2E_k) - 1/2\,]$. Since $E_{k=0} \;\to\; 0$ for $T \;\to\; T_0'$, we see that $n_{k=0}$ diverges at $T = T_0'$. The quasiparticles start to condense in the $\vec{k} = 0$ state in very much the same way as in the unpaired state when real particles condense as the chemical potential approaches $m_\pi$ for decreasing temperature. The interesting point in the paired case is that the condensation of quasiparticles happens at $\mu < m_\pi$, since it is counterbalanced by the finite value of $\delta$.

The region between the solid lines in Fig. 4 summarizes the domain of non–vanishing $\delta$ in the $\mu$–$T$ plane. We see that the temperature $T_0'$ – where the gap closes up the excitation spectrum to zero at $k = 0$, and quasiparticles begin to condense as singles – is very close to the $T_0$ (dot–dashed line) where the pole in the $2\pi$ propagator ceases to exist. The entire $\mu$–$T$ domain of non–trivial solutions to the gap equation is therefore intimately linked to the region where a pole exists in the $2\pi$ propagator.

In summary, we have shown that finite temperature induces real poles in the $2\pi$ propagator, even for situations where there is no $2\pi$ bound state in free space. The situation is analogous to the Cooper pole of fermion systems, and we therefore studied the corresponding bosonic "gap" equation. This equation has non–trivial solutions in a certain domain of the $\mu$–$T$ plane. Such a region always exists, even in the limit of infinitesimally weak attraction. This is different from the $T = 0$ case discussed by Saint James and Nozières [12], where a nontrivial solution to the gap equation only exists when there is a two boson bound state in free space. Our study has to be considered preliminary. The final aim will be to obtain an equation of state for a hot pion gas within a Bruckner–Hartree–Fock–Bogoliubov approach. One of the major questions in this regard is how to avoid a collapse, since $\pi$–$\pi$ interactions are primarily attractive with no need for a hard core repulsion down to very small distances. Also, the subtle question of single boson versus pair condensation must be addressed (see Ref. [12] and references therein). In view of the strongly off–shell

nature of the problem, the $\pi$–$\pi$ interaction should be chosen to be more realistic than is the case here. Work on these topics is in progress.

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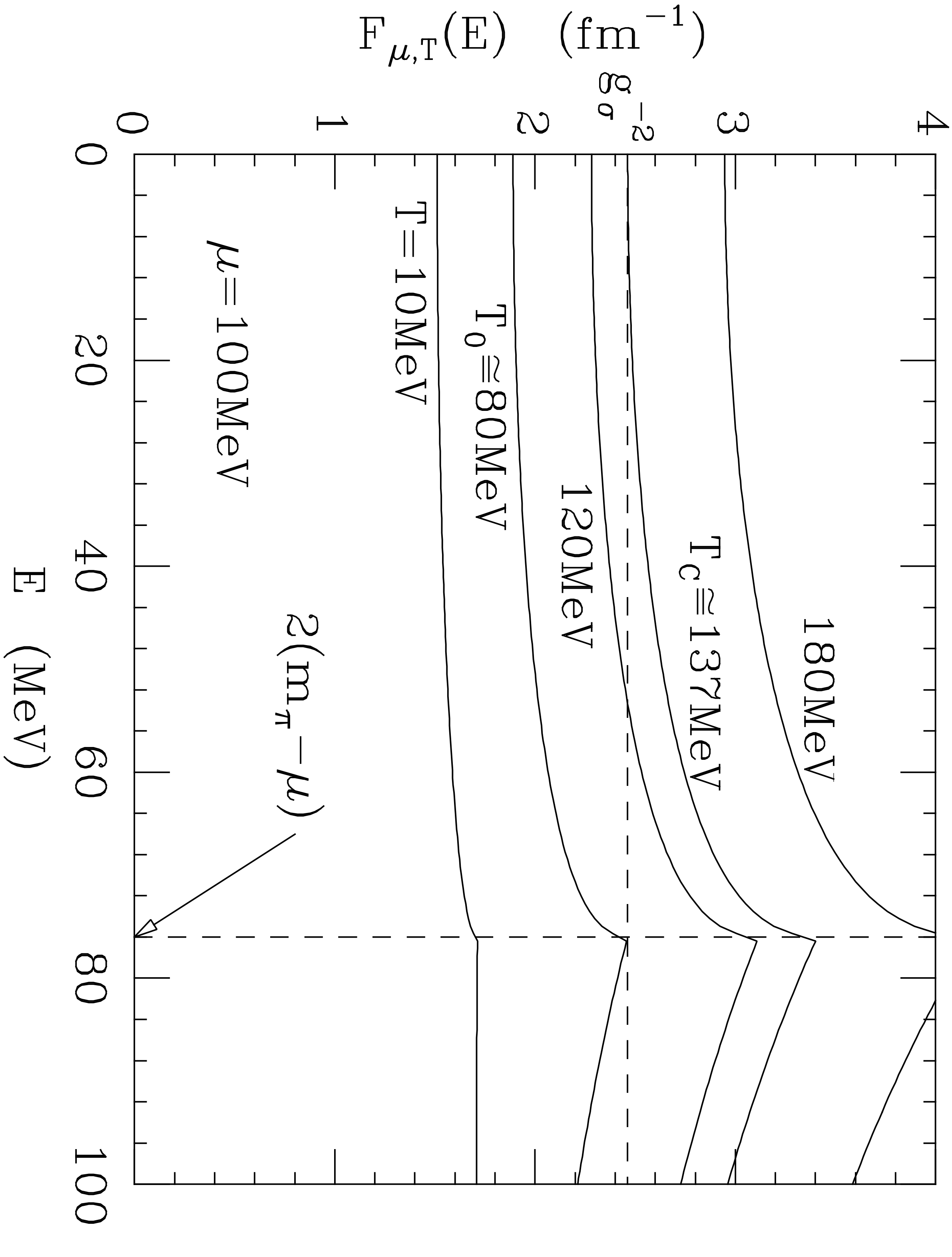
# Figure Captions

**Fig.1** The function $F_{\mu,T}$ of Eq. (9) versus energy, for a fixed chemical potential of $\mu = 100$ MeV and several temperatures (solid lines). Intersections of the solid lines with the horizontal dashed line ($1/g_\sigma^2$) correspond to poles in the in–medium $\sigma$ channel $2\pi$ propagator.

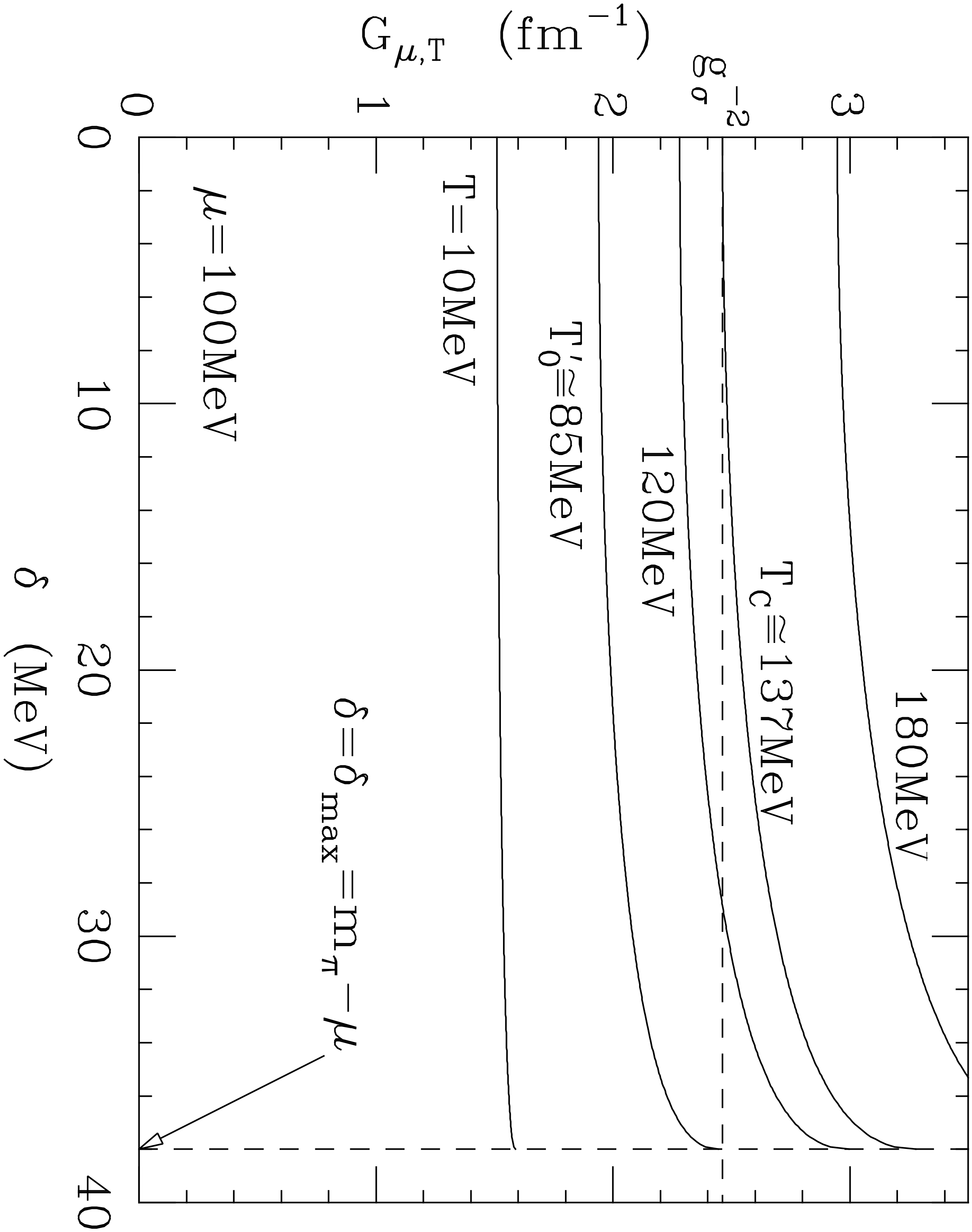
**Fig.2** The function $G_{\mu,T}$ of Eq. (9) versus the gap strength $\delta$, for $\mu = 100$ MeV and several temperatures (solid lines). Intersections of the solid lines with the horizontal dashed line ($1/g_\sigma^2$) correspond to solutions of the gap equation (8). At the lower critical temperature $T_0'$ the quasiparticle energy vanishes ($\delta = \delta_{\rm max}$), while the gap vanishes at the upper limit $T_c$.

**Fig.3** The gap strength versus $m_\pi - \mu$ at several temperatures.

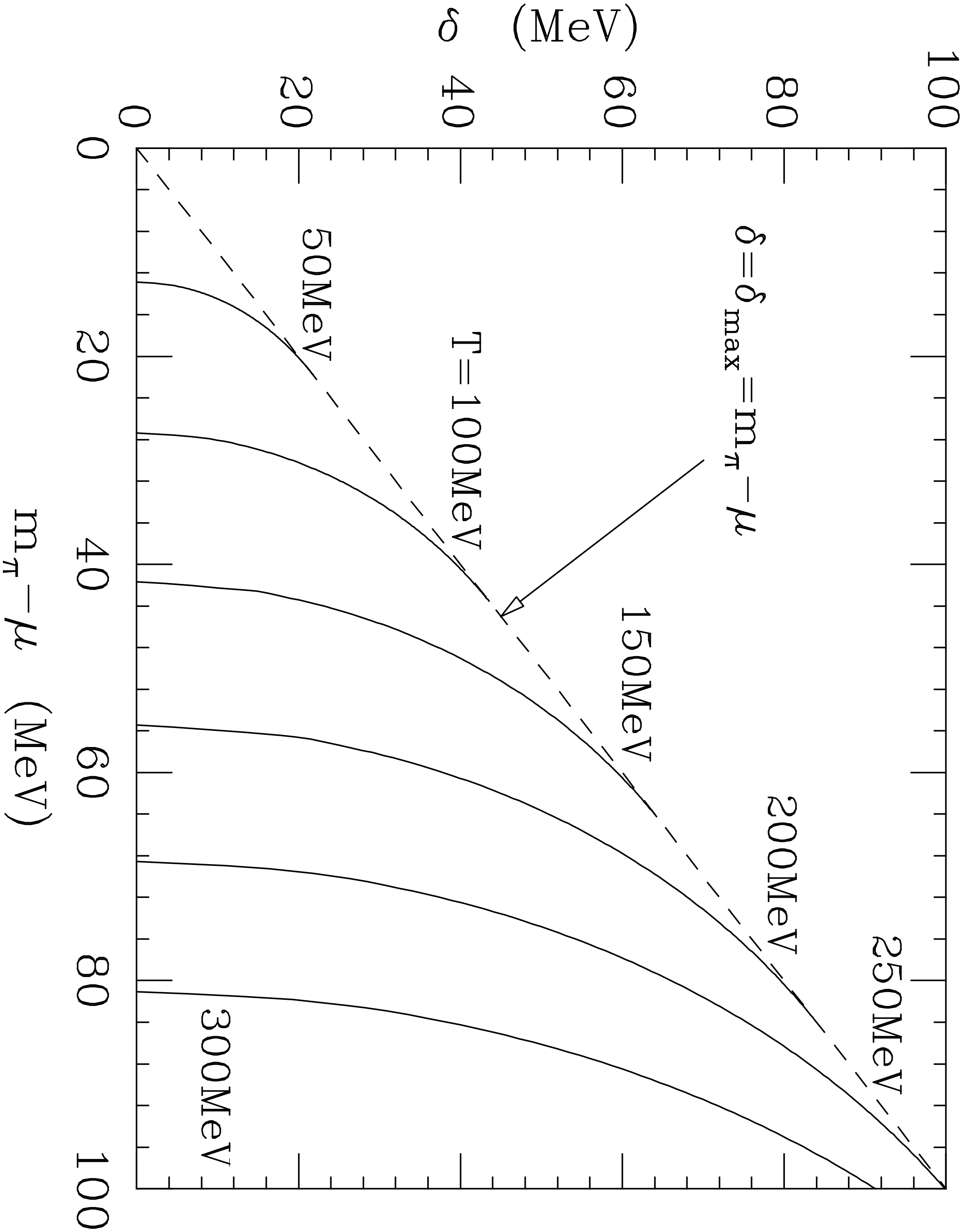
**Fig.4** The area between the solid lines corresponds to the domain in the $\mu$–$T$ plane for which the gap equation (8) has non–trivial solutions. The $2\pi$ propagator has a pole in the region between $T_0$ (dot–dashed line, $E_{\rm pole} = 2(m_\pi - \mu)$) and $T_c$ ($E_{\rm pole} = 0$).



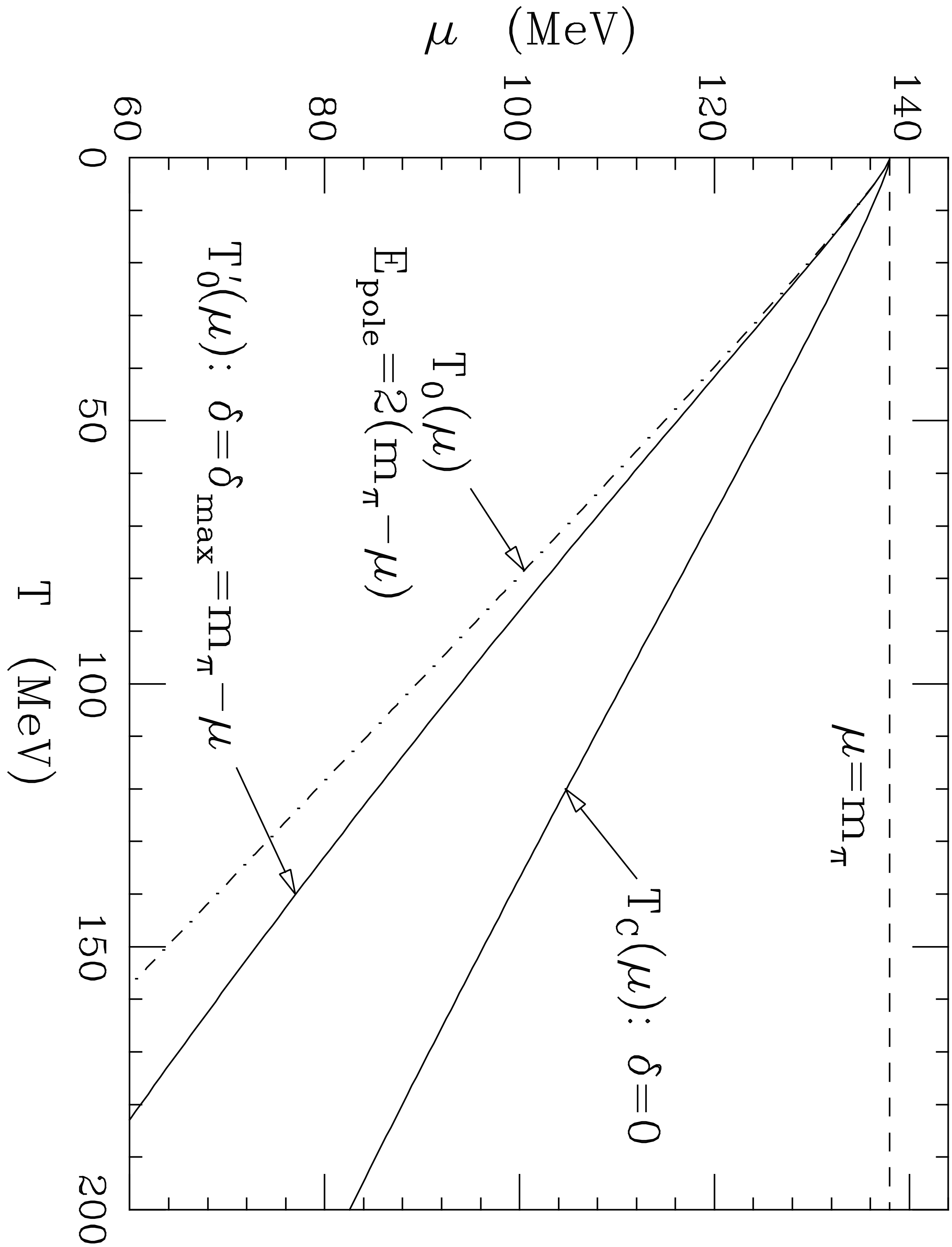

Chanfray et al, Fig.1




Chanfray et al, Fig.2




Chanfray et al, Fig.3




Chanfray et al, Fig.4